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Zero Forcing, Linear and Quantum Controllability for Systems Evolving on Networks

Daniel Burgarth, Domenico D'Alessandro, Leslie Hogben, Simone Severini, and Michael Young

***Abstract*—We study the dynamics of systems on networks from a linear algebraic perspective. The control theoretic concept of *controllability* describes the set of states that can be reached for these systems. Our main result says that controllability in the quantum sense, expressed by the Lie algebra rank condition, and controllability in the sense of linear systems, expressed by the controllability matrix rank condition, are equivalent conditions. We also investigate how the graph theoretic concept of a zero forcing set impacts the controllability property; if a set of vertices is a zero forcing set, the associated dynamical system is controllable. These results open up the possibility of further exploiting the analogy between networks, linear control systems theory, and quantum systems Lie algebraic theory. This study is motivated by several quantum systems currently under study, including continuous quantum walks modeling transport phenomena.**

***Index Terms*—Control, graph, Lie algebra, quantum system, walk matrix, zero forcing.**

I. Introduction

This technical note deals with several concepts from quantum and classical (linear) control theory, linear algebra, and graph theory. In the context of dynamics and control of systems on networks, it establishes connections between a notion in graph theory (zero forcing) and the concepts of quantum and classical controllability in control theory. We review these different concepts before we introduce the technical content of the technical note and give physical motivation for our study.

A. Background

For a dynamical system with a *control input*, the property of *controllability* describes to what extent one can go from one state to another with the evolution corresponding to an appropriate choice of the control. If all the possible state transfers can be obtained within a natural set (the phase space), then the system is said to be *controllable*.

For several classes of systems, controllability has been described in detail and controllability tests are known. In particular, for *linear systems*

$$\dot{\mathbf{x}} = A\mathbf{x} + \sum_{j=1}^{s} \mathbf{b}_j u_j \tag{1}$$

$A \in \mathbb{R}^{n\times n}$, $\mathbf{b}_j \in \mathbb{R}^n$, $j = 1, 2, \ldots, s$, where both the state $\mathbf{x} \in \mathbb{R}^n$ and the control functions $u_j = u_j(t)$ enter the right hand side linearly, several equivalent conditions of controllability are known. The classical controllability condition (see, e.g., [17]) says the system (1) is controllable if and only if the $n \times (ns)$ matrix $\widetilde{W}(A, B) := [\mathbf{b}_1, A\mathbf{b}_1, \ldots, A^{n-1}\mathbf{b}_1, \ldots, \mathbf{b}_s, A\mathbf{b}_s, \ldots, A^{n-1}\mathbf{b}_s]$, has full rank n, where $B := [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \mathbf{b}_s]$ (note that $\widetilde{W}(A, B)$ is obtained from the *controllability matrix* $\mathcal{C}(A, B) = [B, AB, \ldots, A^{n-1}B]$ by a permutation of the columns). In this case, for any prescribed state transfer $\mathbf{x}_0 \to \mathbf{x}_1 (\in \mathbb{R}^n)$ and interval $[0, T]$, there exists a control $\mathbf{u}(t) = [u_1, \ldots, u_s]^T$ such that the corresponding solution $\mathbf{x}(t)$ of (1) satisfies $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_1$.

For quantum mechanical systems that are closed (i.e., not interacting with the environment) and finite dimensional, one considers the *Schrödinger equation*

$$i\frac{d}{dt}|\psi\rangle = H(\mathbf{u})|\psi\rangle \tag{2}$$

where $|\psi\rangle \in \mathbb{C}^n$ is the quantum state and the Hamiltonian matrix $H = H(\mathbf{u})$ is Hermitian and depends on a control $\mathbf{u} = \mathbf{u}(t)$ which in some cases can be assumed to be a switch between different Hamiltonians. If (2) is a system linear in the state $|\psi\rangle$, the solution of (2) is $|\psi(t)\rangle = X(t)|\psi(0)\rangle$ where $X = X(t)$ is the solution of the *Schrödinger matrix equation*

$$i\dot{X} = H(\mathbf{u})X \tag{3}$$

with initial condition equal to the $n \times n$ identity matrix I_n. Since $H = H(\mathbf{u})$ is Hermitian for every value of $\mathbf{u}$ and therefore $-iH$ is skew-Hermitian, the solution of (3) is forced to be unitary at every time t. In this context, the system is called completely controllable if for any unitary matrix X_f in $SU(n)$[1] there exists a control function $\mathbf{u} = \mathbf{u}(t)$ and an interval $[0, T]$ such that the corresponding solution $X = X(t)$ of (3) satisfies $X(0) = I_n$ and $X(T) = X_f$.

At the beginning of the development of the theory of quantum control, it was realized (see e.g., [13]) that system (3) has a structure familiar in geometric control theory [15] and therefore controllability conditions developed there can be directly applied. In particular, the Lie algebra rank condition [16] says that a necessary and sufficient condition for complete controllability of system (3) is that the Lie algebra generated by the matrices $\{iH(\mathbf{u})\}$ (as $\mathbf{u}$ varies in the set of admissible values for the control) is $su(n)$ or $u(n)$.[2] This has given rise to a comprehensive approach to quantum control based on the application of techniques of Lie algebras and Lie group theory [9].

In recent years, there has been considerable interest in the study of control systems, both classical and quantum, which are naturally modeled on networks. One direction in this research is provided by the literature on '*consensus*' problems where interconnected systems, e.g., in robotics [4], which interact in various ways, cooperate to reach a certain desired collective behavior [21], [22]. Often one tries to relate the controllability of systems on networks to topological or graph theoretic properties of the network. For quantum systems, the nodes of the network may represent energy levels or particles which are interacting with each other. For these systems, the application of the Lie algebra rank condition to determine controllability can become cumbersome

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[1]Following standard notation, $SU(n)$ is the special unitary group, i.e., the matrix group of $n \times n$ unitary matrices having determinant 1.

[2]Following standard notation, $u(n)$ is the Lie algebra of $n \times n$ skew-Hermitian matrices and $su(n)$ is the Lie algebra of $n \times n$ skew-Hermitian matrices with zero trace.

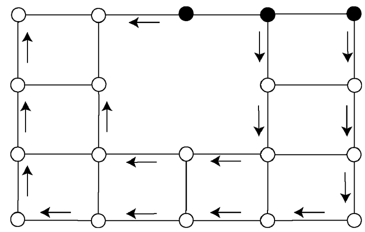

Fig. 1. Zero forcing set and the process by which it can infect all vertices.

and subject to errors when the dimension of the system becomes large. It is preferable to have criteria based on graph theoretic properties of the network not only because they are typically checked more efficiently but also because they give more insight in the dynamics of the system. Work in this direction has been done in [2], [5]–[7], [23]. In this context, a relevant property of a graph G and a subset S of its vertices is the capability of this set to 'infect' all the vertices of the graph, as explained in the next paragraph.

Every graph discussed is simple (no loops or multiple edges), undirected, and has a finite nonempty vertex set. Consider a graph G and color each of its vertices black or white. A vertex v is said to *infect*, or *force*, a vertex w if v is black, w is white, w is a neighbor of v, and w is the only white neighbor of v. In the case where infection of w has occurred, we change the color of w to black and continue the iterative procedure. The set S is called a *zero forcing set* if this procedure, starting from a graph where only the vertices in S are black, leads to a graph where *all* vertices are black. An example of a zero forcing (infection) process is shown in Fig. 1, indicated by arrows; the set of black vertices is a zero forcing set.

For a real symmetric $n \times n$ matrix $A = [a_{kj}]$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{kj : a_{kj} \neq 0 \text{ and } k \neq j\}$. Observe that $G = \mathcal{G}(A_G) = \mathcal{G}(L_G)$, where A_G and $L_G = D_G - A_G$ denote the adjacency matrix of G and the Laplacian matrix of G, respectively (here D_G is the diagonal matrix of degrees). Zero forcing has been studied in detail in linear algebra, because the size of a minimum zero forcing set of a given graph G, which is called the *zero forcing number* $\mathrm{Z}(G)$, is an upper bound to the maximum nullity (or maximum co-rank) over any field of G [3]; the maximum nullity is taken over all symmetric matrices A such that $\mathcal{G}(A) = G$ (see [10] for background on the problem of determining maximum nullity).

Zero forcing appears then to be a valuable concept in the study of graph-theoretic properties that are captured by generalized adjacency matrices. Indeed, there are important classical parameters introduced with this purpose, e.g., the Colin de Verdière number, the Haemers bound, etc. It has to be remarked that questions about the maximum nullity of a graph are generally difficult problems and the zero forcing number does not constitute an exception: it was shown in [1] that there is no poly-logarithmic approximation algorithm for the zero forcing number.

B. Contribution of the Paper and Physical Motivation

In this technical note, we consider the dynamics of a system defined on a network and relate the above ideas of controllability to each other and to the graph theoretic concept of zero forcing. Abstractly, we consider a graph G and a subset $S = \{j_1, \ldots, j_s\}$ of its vertices $V(G) = \{1, \ldots, n\}$. The dynamics are that of a quantum system (3) where the Hamiltonian is allowed to take the values $\{A, \mathbf{e}_{j_1}\mathbf{e}_{j_1}^T, \ldots, \mathbf{e}_{j_s}\mathbf{e}_{j_s}^T\}$. Here A is the adjacency matrix A_G of G, Laplacian matrix L_G of G, or more generally a real symmetric matrix such that $\mathcal{G}(A) = G$ with all nonzero off-diagonal entries of A having the same sign (which is the typical situation in transport models). The vectors $\{\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_s}\}$ are the characteristic vectors[3] of the vertices in S. In this way, we can associate a linear system (1) with A and $\mathbf{b}_1 = \mathbf{e}_{j_1}, \ldots, \mathbf{b}_s = \mathbf{e}_{j_s}$. The main result of the present technical note says that controllability in the quantum sense, expressed by the Lie algebra rank condition, and controllability in the sense of linear systems, expressed by the controllability matrix rank condition, are equivalent conditions (see Corollary 3.7). Moreover, if the set S (corresponding to $\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_s}$) is a zero forcing set, then these equivalent controllability conditions are true (see Corollary 4.2); the converse is false. Corollary 3.7 generalizes the main result of [12] which considers the case of quantum dynamics switching between the Hamiltonian A and $\mathbf{z}\mathbf{z}^T$, where $\mathbf{z} = \sum_{j \in S} \mathbf{e}_j$, and establishes the connection between controllability (quantum and linear). Corollary 4.2 has also appeared in [7]. As mentioned above, these characterizations avoid lengthy calculations of the Lie algebra generated by a given set of Hamiltonians and replace them with more easily verified graph theoretic and linear algebra tests.

The mathematical setup chosen here reflects a situation commonly encountered in quantum physics, where one has a Hamiltonian H with eigenstates $\mathbf{z}_j$ and an interaction V represented by the matrix $A = \mathbf{z}^T V \mathbf{z}$. In the presence of the interaction, the $\mathbf{z}_j$ are no longer good eigenstates; nevertheless the projectors $\mathbf{z}_j^T \mathbf{z}$ still correspond to the control operations one has on the system, e.g., local control, or spectral control. Hence (3) describes a dynamical situation commonly found in experiments. An important example where the graph of the system influences controllability is given by biological quantum systems, recently discussed in [8]. *Continuous time quantum walks* model transport phenomena in many physical and biological systems; a recent review is given in [20]. Many studies consider this sole Hamiltonian and concern statistical (diffusion) properties of the dynamics. We add here the Hamiltonians $\mathbf{e}_j\mathbf{e}_j^T$ where $\mathbf{e}_j$ is the characteristic vector of a given node of the network and study the nature of the states that the resulting dynamics can achieve, in particular whether an arbitrary (unitary) state transfer can be achieved between the states of the quantum system. The Hamiltonians $\mathbf{e}_j\mathbf{e}_j^T$ model a prescribed energy difference between the corresponding node and all the other nodes of the network, which are assumed to be at the same energy level. Thus the dynamics is the alternating of a diffusion process (modeled by the Hamiltonian A) and a rearrangement of the energies of the various states by selecting one of the states as high energy state and all the other at the same (lower) energy.

Theoretical research in network theory has focused on a number of discrete time, deterministic diffusion processes on graphs. While zero forcing has not been studied in this context, there are two directions of research that are closely related: as it was noted in [1], the threshold model introduced for studying influence in social networks shares with zero forcing certain issues underlying its computational complexity [18]; the model of complex networks controllability proposed in [19] makes a natural use of the controllability matrix rank condition and singles out certain combinatorial properties to determine when the condition is satisfied. The connection between zero forcing and quantum control has been studied in [5]–[7]. Determining whether zero forcing has a place in the metrology of complex networks is a point of further interest.

The technical note is organized as follows. In Section II we introduce notation and give background and basic results concerning Lie algebras that will be used in the following sections. The connection between quantum (Lie algebraic) controllability and the controllability matrix rank criterion for linear systems is established in Section III. There we also prove the converse of the main result of [12]. The relation with the zero forcing property is discussed in Section IV, and Section V contains concluding remarks.

[3]The vector $\mathbf{e}_j$ has the jth entry equal to one and every other entry equal to zero.

II. Lie Algebra Terminology and Preliminary Results

Standard material on Lie algebras can be found in [14]. For $A_1,\dots,A_k \in \mathbb{C}^{n\times n}$, $\langle A_1,\dots,A_k\rangle_{[\cdot,\cdot]}$ denotes the real Lie algebra generated by $A_1,\dots,A_k$ under addition, real scalar multiplication, and the commutator operation. Let $\mathcal{H}_n(\mathbb{R})$ denote the real vector space of symmetric matrices. For $A\in\mathcal{H}_n(\mathbb{R})$, the notation $A=[a_{kj}]$ means for $k<j$ the (k,j) and (j,k) entries of A are both a_{kj}. Observe that $A=[a_{kj}]\in\mathcal{H}_n(\mathbb{R})$ can be expressed as

$$A=\sum_{k=1}^{n} a_{kk}\mathbf{e}_k\mathbf{e}_k^T+\sum_{k<j} a_{kj}\left(\mathbf{e}_k\mathbf{e}_j^T+\mathbf{e}_j\mathbf{e}_k^T\right).$$

The following proposition is well known (a proof appears in [12]). It provides a link between an appropriate Lie algebra of real matrices and the Lie algebra rank condition of quantum control theory, allowing us to work with real matrices only. Recall the Lie algebra consisting of all real $n\times n$ matrices is denoted by $gl(n,\mathbb{R})$, $sl(n,\mathbb{R})$ denotes the Lie algebra of real $n\times n$ matrices with zero trace, $u(n)$ denotes the Lie algebra of all skew-Hermitian (complex) $n\times n$ matrices, and $su(n)$ denotes the Lie algebra of all skew-Hermitian (complex) $n\times n$ matrices with zero trace. All these Lie algebras are considered as vector spaces over the field of real numbers.

Proposition 2.1: For $A_1,\dots,A_k\in\mathcal{H}_n(\mathbb{R})$,

$$\langle A_1,\dots,A_k\rangle_{[\cdot,\cdot]}=gl(n,\mathbb{R}) \iff \langle iA_1,\dots,iA_k\rangle_{[\cdot,\cdot]}=u(n).$$

The next lemma is used in the proof of Theorem 3.6 in the next section.

Lemma 2.2: Let $A,B_1,\dots,B_s\in\mathcal{H}_n(\mathbb{R})$, with $s\geq 1$. Define $\mathcal{L}:=\langle A,B_1,\dots,B_s\rangle_{[\cdot,\cdot]}$ and let $\hat{\mathcal{L}}$ denote the smallest ideal of $\mathcal{L}$ that contains B_i, $i=1,\dots,s$. If $\mathcal{L}=gl(n,\mathbb{R})$ and $\mathrm{tr}B_k\neq 0$ for some B_k, then $\hat{\mathcal{L}}=gl(n,\mathbb{R})$.

Proof: For $n=1$, the result is clear, so assume $n\geq 2$, $\mathcal{L}=gl(n,\mathbb{R})$, and $\mathrm{tr}B_k\neq 0$ for some B_k. Observe that $\mathcal{L}:=\langle A,B_1,\dots,B_s\rangle_{[\cdot,\cdot]}$ is spanned by A and $\hat{\mathcal{L}}$. Since $\mathcal{L}=gl(n,\mathbb{R})$, we have $[gl(n,\mathbb{R}),gl(n,\mathbb{R})]=[\mathrm{span}(A)+\hat{\mathcal{L}},\mathrm{span}(A)+\hat{\mathcal{L}}]\subseteq\hat{\mathcal{L}}$. It is known that $[gl(n,\mathbb{R}),gl(n,\mathbb{R})]=sl(n,\mathbb{R})$, because $[gl(n,\mathbb{R}),gl(n,\mathbb{R})]$ is a nonzero ideal in $sl(n,\mathbb{R})$ and $sl(n,\mathbb{R})$ is a simple Lie algebra. Since $\dim sl(n,\mathbb{R})=n^2-1$ and $B_k\notin sl(n,\mathbb{R})$, $\dim\hat{\mathcal{L}}\geq n^2$. Thus $\hat{\mathcal{L}}=gl(n,\mathbb{R})$. ■

The next lemma is used in the proof of Theorem 3.1 in the next section. Let $\mathcal{L}$ be a Lie algebra, $A\in\mathcal{L}$, and let $\mathcal{K}$ be a subspace of $\mathcal{L}$. Recall that the operation ad_A is defined as $ad_A(B):=[A,B]$, and the *normalizer* of $\mathcal{K}$ is $N_{\mathcal{L}}(\mathcal{K})=\{A:[A,B]\in\mathcal{K}\text{ for all }B\in\mathcal{K}\}$. It follows from the Jacobi identity that $N_{\mathcal{L}}(\mathcal{K})$ is a subalgebra of $\mathcal{K}$ [14, p. 7].

Lemma 2.3: Let $A,L\in\mathcal{H}_n(\mathbb{R})$. Assume $\langle iA,iL\rangle_{[\cdot,\cdot]}=u(n)$ and define

$$\mathcal{S}:=\mathrm{span}\left(\{ad_{iA}^{k_1}ad_{iL}^{k_2}\cdots ad_{iA}^{k_{s-1}}ad_{iL}^{k_s}[iA,iL]\}\right) \tag{4}$$

where s and $k_1,\dots,k_s$ are nonnegative integers. Then $\mathcal{S}=su(n)$.

Proof: First note that $[iA,iL]\neq 0$ because we have assumed that iA and iL generate $u(n)$. Clearly $iA,iL\in N_{u(n)}(\mathcal{S})$. Since $N_{u(n)}(\mathcal{S})$ is a subalgebra of $u(n)$ and iA and iL generate $u(n)$, $N_{u(n)}(\mathcal{S})=u(n)$. Thus $\mathcal{S}$ is an ideal of $u(n)$. Notice that $\mathcal{S}\subseteq su(n)$ since $[iA,iL]$ is skew-Hermitian with zero trace and iA and iL are skew-Hermitian. Since $\mathcal{S}$ is an ideal of $u(n)$, $\mathcal{S}$ is an ideal of $su(n)$, and $\mathcal{S}\neq\{0\}$. Since $su(n)$ is a simple Lie algebra, by definition it has only the trivial ideals $\{0\}$ and $su(n)$. Therefore $\mathcal{S}=su(n)$. ■

For $A\in\mathcal{H}_n(\mathbb{R})$ and $Z=\{\mathbf{z}_1,\dots,\mathbf{z}_s\}\subset\mathbb{R}^n$, the *real Lie algebra generated by* A *and* Z is defined as

$$\mathcal{L}(A,Z):=\left\langle A,\mathbf{z}_1\mathbf{z}_1^T,\dots,\mathbf{z}_s\mathbf{z}_s^T\right\rangle_{[\cdot,\cdot]}. \tag{5}$$

III. Controllability and Walk Matrices

In this section, we show that controllability in the quantum sense, expressed by the Lie algebra rank condition, and controllability in the sense of linear systems, expressed by the controllability matrix rank condition, are equivalent.

For $A\in\mathcal{H}_n(\mathbb{R})$ and $Z=\{\mathbf{z}_1,\dots,\mathbf{z}_s\}\subset\mathbb{R}^n$, the *(extended) walk matrix* of A and Z is the $n\times(ns)$ real matrix

$$\widetilde{W}(A,Z):=[\mathbf{z}_1,A\mathbf{z}_1,\dots,A^{n-1}\mathbf{z}_1,\dots,\mathbf{z}_s,A\mathbf{z}_s,\dots,A^{n-1}\mathbf{z}_s]. \tag{6}$$

A special case is when $Z=Z_S:=\{\mathbf{e}_j:j\in S\}$ for some subset $S\subseteq V(G)$ for a graph G and A is the adjacency matrix A_G of the graph; here the relevant walk matrix is $\widetilde{W}(A_G,Z_S)$. The (extended) walk matrix represents the Hamiltonian in the system, expressed in a specific basis. This basis often corresponds to the "local" states or those which can be manipulated within an experiment, rather than the eigenstates (in which the graph would always be trivial). An example would be a Heisenberg Hamiltonian within the first excitation sector, for which the local states represent local spin flips. Then the controls are local states that can be affected, for instance by a time-dependent magnetic field.

For $s=1$ the connection between the walk matrix $\widetilde{W}(A,Z)$ in (6) and the Lie algebra $\mathcal{L}(A,Z)$ in (5) was studied in [12]. It was shown [12, Lemma 1] that $\mathrm{rank}\widetilde{W}(A,\{\mathbf{z}\})=n$ implies $\mathcal{L}(A,\{\mathbf{z}\})=gl(n,\mathbb{R})$ (although the result is stated for the adjacency matrix and a 0,1-vector, the proof remains valid more generally for $A\in\mathcal{H}_n(\mathbb{R})$ and $\mathbf{z}\in\mathbb{R}^n$), or equivalently, $\langle iA,i\mathbf{z}\mathbf{z}^T\rangle_{[\cdot,\cdot]}=u(n)$ (cf. Proposition 2.1). The next theorem states that the converse is true.

Theorem 3.1: Consider a matrix A in $\mathcal{H}_n(\mathbb{R})$ and a vector $\mathbf{z}\in\mathbb{R}^{n\times n}$. Then, $\langle iA,i\mathbf{z}\mathbf{z}^T\rangle_{[\cdot,\cdot]}=u(n)$ (or equivalently $\mathcal{L}(A,\{\mathbf{z}\})=gl(n,\mathbb{R})$) implies that $\mathrm{rank}\widetilde{W}(A,\{\mathbf{z}\})=n$.

Proof: The equivalence of the hypotheses is justified by Proposition 2.1. The result is clear if $n=1$, so assume $n\geq 2$. We use a contradiction argument. Assume the rank of the walk matrix $\widetilde{W}(A,\{\mathbf{z}\})$ is less than n but $\langle iA,iL\rangle_{[\cdot,\cdot]}=u(n)$, where $L:=\mathbf{z}\mathbf{z}^T$. There exists a nonzero vector $\mathbf{x}\in\mathbb{R}^n$ such that $\mathbf{x}^*\widetilde{W}(A,\{\mathbf{z}\})=0$. Consider the rank 1 matrix $D:=\mathbf{x}\mathbf{x}^*$. We claim that D commutes with every matrix in $\mathcal{S}$, where $\mathcal{S}$ is as in (4). To see this, notice that from (4), all elements in $\mathcal{S}$ are linear combinations of monomials of the form $M=A^{k_1}L^{k_2}A^{k_3}\cdots L^{k_{p-1}}A^{k_p}$, for some $p\geq 1$, $k_j\geq 0$, and L appearing at least once with exponent greater than zero. When multiplying D with M, with D on the left, write M as $A^{k_1}LY$ for some matrix Y, so we have

$$DM=DA^{k_1}LY=\mathbf{x}\mathbf{x}^*A^{k_1}\mathbf{z}\mathbf{z}^*Y=0 \tag{7}$$

which follows immediately from the condition $\mathbf{x}^*\widetilde{W}(A,\{\mathbf{z}\})=0$ for $n-1\geq k_1\geq 0$, and by using the Cayley-Hamilton theorem for $k_1\geq n$. Analogously, when multiplying D on the right of M, we write M as QLA^{k_p}, for some matrix Q, and we have

$$MD=QLA^{k_p}D=Q\mathbf{z}\mathbf{z}^*A^{k_p}\mathbf{x}\mathbf{x}^*=0 \tag{8}$$

since $\mathbf{x}^*A^{k_p}\mathbf{z}=0$ also implies $\mathbf{z}^*A^{k_p}\mathbf{x}=0$. Therefore D commutes with all elements of $\mathcal{S}$.

Observe that since $su(n)$ is simple, $su(n)$ is an irreducible representation of $su(n)$. Therefore, since D commutes with all elements of $\mathcal{S}$, it follows from Schur's Lemma that D must be a scalar multiple of the identity [14, p. 26]. However this is not possible since D has rank 1. This gives the desired contradiction and thus completes the proof. ■

We study the generalization of this result to multiple vectors ($s \geq 1$) but for matrices A and vectors $\mathbf{z}_1, \ldots, \mathbf{z}_s$ related to a connected graph G. In particular, $\mathcal{G}(A) = G$, all nonzero off-diagonal entries of A have the same sign, and $\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_s}$ will be the characteristic vectors associated to a subset S of the vertices. In the next section we will relate this to the zero forcing property of the set S. In the context of graphs, it is important to consider multiple vectors because if G is a graph and $\mathrm{rank} A_G \leq |G| - 2$, then $\mathrm{rank}\, \widetilde{W}(A_G, \{\mathbf{z}\}) < n$ for any one vector $\mathbf{z}$. On the other hand we will see that if S is a zero forcing set for G and $\mathcal{G}(A) = G$, then $\mathcal{L}(A, \{\mathbf{e}_j : j \in S\}) = gl(n, \mathbb{R})$ (see Theorem 4.1 below).

The next definition extends the definition given in [11] (and implicitly in [12]) of an associative algebra that links the walk matrix and controllability. For $A \in \mathcal{H}_n(\mathbb{R})$ and $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_s\} \subset \mathbb{R}^n$, define $P(A, Z) := \{A^m \mathbf{z}_k \mathbf{z}_j^T A^\ell : 1 \leq k, j \leq s, 0 \leq m, \ell \leq n-1\}$.

Remark 3.2: For $A \in \mathcal{H}_n(\mathbb{R})$ and $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_s\} \subset \mathbb{R}^n$, the associative algebra generated by $P(A, Z)$ is equal to $\mathrm{span} P(A, Z)$, because

$$\left(A^m \mathbf{z}_k \mathbf{z}_j^T A^\ell\right)\left(A^g \mathbf{z}_p \mathbf{z}_q^T A^h\right) = \left(\mathbf{z}_j^T A^{\ell+g} \mathbf{z}_p\right) A^m \mathbf{z}_k \mathbf{z}_q^T A^h$$

and $\mathbf{z}_j^T A^{\ell+g} \mathbf{z}_p \in \mathbb{R}$.

Lemma 3.3: For $A \in \mathcal{H}_n(\mathbb{R})$ and $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_s\} \subset \mathbb{R}^n$, $\mathrm{rank}\widetilde{W}(A, Z) = n$ if and only if $\mathrm{span} P(A, Z) = \mathbb{R}^{n \times n}$.[4]

Proof: Clearly $\mathrm{rank}\widetilde{W}(A, Z) = n$ if and only if $\mathrm{range}\widetilde{W}(A, Z) = \mathbb{R}^n$. First assume $\mathrm{rank}\widetilde{W}(A, Z) = n$. For any matrix $M \in \mathbb{R}^{n \times n}$ with $\mathrm{rank} M = r$, there exist vectors $\mathbf{x}^{(q)}, \mathbf{y}^{(q)}$, $q = 1, \ldots r$, such that $M = \sum_{q=1}^r \mathbf{x}^{(q)} \mathbf{y}^{(q)^T}$. Since $\mathrm{range}\widetilde{W}(A, Z) = \mathbb{R}^n$, each $\mathbf{x}^{(q)}$ is expressible as a linear combination of the columns of $\widetilde{W}(A, Z)$, i.e., as a linear combination of vectors of the form $A^m \mathbf{z}_k$, and similarly for $\mathbf{y}^{(q)}$. Thus each $\mathbf{x}^{(q)} \mathbf{y}^{(q)^T}$, and hence M, is expressible as a linear combination of $A^m \mathbf{z}_k \mathbf{z}_j^T A^\ell$. Thus the matrices of the form $A^m \mathbf{z}_k \mathbf{z}_j^T A^\ell$ span $\mathbb{R}^{n \times n}$.

For the converse, observe that if $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_r\}$ is a basis for $\mathrm{range}\widetilde{W}(A, Z)$, then $\mathrm{span} P(A, Z) = \mathrm{span}(\{\mathbf{b}_k \mathbf{b}_j^T : 1 \leq k, j \leq r\})$. If $n > r = \mathrm{rank}\widetilde{W}(A, Z)$, then $\dim \mathrm{span} P(A, Z) \leq r^2 < n^2 = \dim \mathbb{R}^{n \times n}$, so the matrices in $P(A, Z)$ cannot span $\mathbb{R}^{n \times n}$. ■

The *distance* between two distinct vertices u and v of a connected graph G, denoted by $d(u, v)$ is the minimum number of edges in a path from u to v.

Lemma 3.4: Let $A \in \mathcal{H}_n(\mathbb{R})$ such that $\mathcal{G}(A)$ is connected and all nonzero off-diagonal entries of A have the same sign. If $k, j \in \{1, \ldots, n\}$ and $k \neq j$, then $(A^{d(k,j)})_{kj} \neq 0$.

Proof: Let $d := d(k, j)$. The entry $(A^d)_{kj}$ is a sum of terms each of which is the product of d nonzero entries of A. Since d is the distance between k and j, only off-diagonal entries can appear in this product. Thus every term has the same sign and $(A^d)_{kj} \neq 0$. ■

Lemma 3.5: Let $A \in \mathcal{H}_n(\mathbb{R})$ be such that $\mathcal{G}(A)$ is connected and all nonzero off-diagonal entries of A have the same sign. Let $S \subseteq \{1, \ldots, n\}$ and $Z = \{\mathbf{e}_j : j \in S\}$. Then $\mathrm{span} P(A, Z) \subseteq \mathcal{L}(A, Z)$.

Proof: The proof of Lemma 1 in [12] shows that for any real symmetric matrix A and vector $\mathbf{z}$, $A^m \mathbf{z}\mathbf{z}^T A^\ell \in \mathcal{L}(A, \{\mathbf{z}\})$ for all $m, \ell \in \{0, \ldots, n-1\}$. Applying this, we obtain that $A^m \mathbf{e}_j \mathbf{e}_j^T A^\ell \in \mathcal{L}(A, Z)$ for all $j \in \{1, \ldots, s\}, m, \ell \in \{0, \ldots, n-1\}$. The result will follow if we are able to show that $A^m \mathbf{e}_k \mathbf{e}_j^T A^\ell \in \mathcal{L}(A, Z)$ for all $k, j \in \{1, \ldots, s\}, m, \ell \in \{0, \ldots, n-1\}$, with k different from j.

Consider the distance $d := d(k, j)$ between the nodes k and j in $\mathcal{G}(A)$, which is $\leq n-1$ because $\mathcal{G}(A)$ is connected. From the fact that both $\mathbf{e}_k \mathbf{e}_k^T$ and $A^d \mathbf{e}_j \mathbf{e}_j^T$ are in $\mathcal{L}(A, Z)$, we have in $\mathcal{L}(A, Z)$, $[\mathbf{e}_k \mathbf{e}_k^T, A^d \mathbf{e}_j \mathbf{e}_j^T] = \mathbf{e}_k \mathbf{e}_k^T A^d \mathbf{e}_j \mathbf{e}_j^T - A^d \mathbf{e}_j \mathbf{e}_j^T \mathbf{e}_k \mathbf{e}_k^T = (\mathbf{e}_k^T A^d \mathbf{e}_j) \mathbf{e}_k \mathbf{e}_j^T$. It follows from Lemma 3.4 that $\mathbf{e}_k^T A^d \mathbf{e}_j = (A^d)_{kj} \neq 0$, and so $\mathbf{e}_k \mathbf{e}_j^T \in \mathcal{L}(A, Z)$. Then

$$\begin{aligned}\left[A^m \mathbf{e}_k \mathbf{e}_k^T, \mathbf{e}_k \mathbf{e}_j^T\right] &= A^m \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_k \mathbf{e}_j^T - \mathbf{e}_k \mathbf{e}_j^T A^m \mathbf{e}_k \mathbf{e}_k^T \\ &= A^m \mathbf{e}_k \mathbf{e}_j^T - \left(\mathbf{e}_j^T A^m \mathbf{e}_k\right) \mathbf{e}_k \mathbf{e}_k^T.\end{aligned}$$

So, $A^m \mathbf{e}_k \mathbf{e}_j^T \in \mathcal{L}(A, Z)$. Similarly, $\mathbf{e}_k \mathbf{e}_j^T A^\ell \in \mathcal{L}(A, Z)$. Finally

$$\begin{aligned}&\left[A^m \mathbf{e}_k \mathbf{e}_k^T, \mathbf{e}_k \mathbf{e}_j^T A^\ell\right] \\ &= A^m \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_k \mathbf{e}_j^T A^\ell - \mathbf{e}_k \mathbf{e}_j^T A^{m+\ell} \mathbf{e}_k \mathbf{e}_k^T \\ &= A^m \mathbf{e}_k \mathbf{e}_j^T A^\ell - \left(\mathbf{e}_j^T A^{m+\ell} \mathbf{e}_k\right) \mathbf{e}_k \mathbf{e}_k^T.\end{aligned}$$

So, $A^m \mathbf{e}_k \mathbf{e}_j^T A^\ell \in \mathcal{L}(A, Z)$. ■

The following theorem establishes the connection between quantum Lie algebraic controllability and the rank condition for an (extended) walk matrix modeled on a graph.

Theorem 3.6: Let $A \in \mathcal{H}_n(\mathbb{R})$ such that $\mathcal{G}(A)$ is connected and all the nonzero off-diagonal elements of A have the same sign. Let $S \subseteq \{1, \ldots, n\}$ and $Z = \{\mathbf{e}_j : j \in S\}$. Then $\mathrm{rank}\widetilde{W}(A, Z) = n$ if and only if $\mathcal{L}(A, Z) = gl(n, \mathbb{R})$.

Proof: By Lemma 3.3, $\mathrm{span} P(A, Z) = \mathbb{R}^{n \times n}$ if and only if $\mathrm{rank}\widetilde{W}(A, Z) = n$, so it suffices to show that $\mathrm{span} P(A, Z) = \mathbb{R}^{n \times n}$ if and only if $\mathcal{L}(A, Z) = gl(n, \mathbb{R})$. By Lemma 3.5, $\mathrm{span} P(A, Z) \subseteq \mathcal{L}(A, Z)$, so $\mathrm{span} P(A, Z) = \mathbb{R}^{n \times n}$ implies $\mathcal{L}(A, Z) = gl(n, \mathbb{R})$ For the converse, assume $\mathcal{L}(A, Z) = gl(n, \mathbb{R})$. Then, by Lemma 2.2, $\hat{\mathcal{L}} = gl(n, \mathbb{R})$, where $\hat{\mathcal{L}}$ is the smallest ideal of $\mathcal{L}(A, Z)$ that contains $\mathbf{e}_j \mathbf{e}_j^T$ for all $j \in S$. It is clear that $\hat{\mathcal{L}} \subseteq \mathrm{span} P(A, Z)$, so $\mathrm{span} P(A, Z) = \mathbb{R}^{n \times n}$. ■

Corollary 3.7: Let $A \in \mathcal{H}_n(\mathbb{R})$ such that $\mathcal{G}(A)$ is connected and all the nonzero off-diagonal elements of A have the same sign, and let $S \subseteq \{1, \ldots, n\}$. Then $\mathrm{rank}\widetilde{W}(A, \{\mathbf{e}_j : j \in S\}) = n$ if and only if $\langle iA, \{i\mathbf{e}_j \mathbf{e}_j^T : j \in S\}\rangle_{[\cdot,\cdot]} = u(n)$, i.e., the quantum system associated with the Hamiltonians iA and $i\mathbf{e}_j \mathbf{e}_j^T$, $j = 1, \ldots, s$, is controllable.

Observe that for any connected graph G, the adjacency matrix A_G and the Laplacian matrix L_G satisfy the hypotheses of Theorem 3.6 and Corollary 3.7.

The result of [12] for the case $s = 1$ showing that $\mathrm{rank}\widetilde{W}(A, \{\mathbf{z}\}) = n$ implies $\langle iA, i\mathbf{z}\mathbf{z}^T\rangle_{[\cdot,\cdot]} = u(n)$, (and the converse proved in Theorem 3.1 in this technical note) were proved in reference to systems on graphs. The proofs however go through for an arbitrary symmetric matrix A and vector $\mathbf{z}$. It is natural to ask whether the conditions on the matrix A that we have used in Theorem 3.6 are really necessary. To this purpose, we can observe that the result is not true if we give up either of the hypotheses that 1) $\mathcal{G}(A)$ is connected or 2) the off-diagonal entries of A have the same sign, as shown in the next two examples.

Example 3.8: To see the necessity of assuming that $\mathcal{G}(A)$ is connected, consider a block diagonal matrix $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with A_1 and A_2 symmetric matrices of dimensions n_1 and n_2, respectively, with $n_1 + n_2 = n$, and let $\mathbf{z}_i \in \mathbb{R}^{n_i}$, $i = 1, 2$ such that the matrices $\widetilde{W}(A_1, \{\mathbf{z}_1\})$ and $\widetilde{W}(A_2, \{\mathbf{z}_2\})$ have ranks n_1 and n_2, respectively. Define $\hat{\mathbf{z}}_1 := [\mathbf{z}_1^T, 0^T]^T$ and $\hat{\mathbf{z}}_2 := [0^T, \mathbf{z}_1^T]^T$. Then the walk matrix $\widetilde{W}(A, \{\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2\})$ has rank n, but the Lie algebra generated by $A, \hat{\mathbf{z}}_1 \hat{\mathbf{z}}_1^T$, and $\hat{\mathbf{z}}_2 \hat{\mathbf{z}}_2^T$ contains only block diagonal matrices.

Example 3.9: To see the necessity of assuming that all nonzero off-diagonal entries of A have the same sign, consider $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, and $Z = \{\mathbf{e}_1, \mathbf{e}_3\}$. It is straightforward

[4]As a vector space, $\mathbb{R}^{n \times n}$ is the same as $gl(n, \mathbb{R})$. We use the latter notation to stress the Lie algebra structure on $gl(n, \mathbb{R})$.

to verify that the walk matrix $\widetilde{W}(A, \{\mathbf{e}_1, \mathbf{e}_3\})$ has rank 4. However, $\dim\mathcal{L}(A, \{\mathbf{e}_1, \mathbf{e}_3\}) \leq 8$, as can be seen as follows. Let

$$\mathcal{L} := \operatorname{span}\left(\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&-1\\0&0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&-1&0\end{bmatrix}, \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{bmatrix}\right).$$

Since $[B, C] \in \mathcal{L}$ for all $B, C \in \mathcal{L}$, $\mathcal{L}$ is a Lie subalgebra of $gl(4, \mathbb{R})$. Clearly, $\dim\mathcal{L} \leq 8$ and $\mathcal{L}(A, \{\mathbf{e}_1, \mathbf{e}_3\}) \subseteq \mathcal{L}$.

IV. Zero Forcing and Controllability

In this section, we discuss the connection between zero forcing and controllability. Theorem 4.1 and its immediate consequence Corollary 4.2 below are proved less formally and with different notation in [7]. The hypothesis that G is connected was omitted there (Example 3.8 with $n_1 = n_2 = 1$ shows the necessity of connectivity). We include the brief proof here for completeness. Corollary 4.2 is much more general than results in [5] because it can be applied to *any* finite dimensional Hamiltonian and does not rely on the tensor product structure in [5] (but the tensor product criterion is more efficient when it applies). The *neighborhood* of $v \in V(G)$ is $N(v) = \{w \in V(G) : w \text{ is adjacent to } v\}$.

Theorem 4.1: Let $A \in \mathcal{H}_n(\mathbb{R})$ such that $\mathcal{G}(A)$ is connected. Let $V := \{1, 2, \ldots, n\}$ be the set of vertices for $\mathcal{G}(A)$, and let $S \subseteq V$ be a zero forcing set of $\mathcal{G}(A)$. Then

$$\mathcal{L}\left(A, \left\{\mathbf{e}_j\mathbf{e}_j^T : j \in S\right\}\right) = gl(n, \mathbb{R}).$$

Proof: Let $\mathcal{L} := \mathcal{L}(A, \{\mathbf{e}_j\mathbf{e}_j^T : j \in S\})$, and let $\tilde{a}_{uv} := a_{uv}$ if $u < v$ and $\tilde{a}_{uv} := a_{vu}$ if $v < u$. If $\mathbf{e}_u\mathbf{e}_u^T, \mathbf{e}_v\mathbf{e}_v^T \in \mathcal{L}$ and u and v are neighbors, then

$$\mathbf{e}_u\mathbf{e}_v^T + \mathbf{e}_v\mathbf{e}_u^T = \frac{1}{\tilde{a}_{uv}}\left[\left[\mathbf{e}_u\mathbf{e}_u^T, A\right], \mathbf{e}_v\mathbf{e}_v^T\right] \in \mathcal{L}. \tag{9}$$

If $\mathbf{e}_u\mathbf{e}_u^T, \mathbf{e}_u\mathbf{e}_v^T + \mathbf{e}_u\mathbf{e}_v^T \in \mathcal{L}$, then

$$\mathbf{e}_u\mathbf{e}_v^T = \frac{1}{2}\left(\left[\mathbf{e}_u\mathbf{e}_u^T, \mathbf{e}_u\mathbf{e}_v^T + \mathbf{e}_v\mathbf{e}_u^T\right] + \mathbf{e}_u\mathbf{e}_v^T + \mathbf{e}_v\mathbf{e}_u^T\right) \in \mathcal{L}. \tag{10}$$

After a (possibly empty) sequence of forces, denote by T the set of currently black vertices, and assume that for all $v \in T$, $\mathbf{e}_v\mathbf{e}_v^T \in \mathcal{L}$. If $T \neq V$, then there is a vertex $u \in T$ that has a unique neighbor w outside T, so for that u and w: $-\sum_{v \in N(u)} \tilde{a}_{uv}(\mathbf{e}_u\mathbf{e}_v^T + \mathbf{e}_v\mathbf{e}_u^T) + 2a_{uu}\mathbf{e}_u\mathbf{e}_u^T = [[\mathbf{e}_u\mathbf{e}_u^T, A], \mathbf{e}_u\mathbf{e}_u^T] \in \mathcal{L}$. Then by (9), $\mathbf{e}_u\mathbf{e}_w^T + \mathbf{e}_w\mathbf{e}_u^T \in \mathcal{L}$. Then by (10), $\mathbf{e}_u\mathbf{e}_w^T, \mathbf{e}_w\mathbf{e}_u^T \in \mathcal{L}$, so $\mathbf{e}_w\mathbf{e}_w^T = [\mathbf{e}_w\mathbf{e}_u^T, \mathbf{e}_u\mathbf{e}_w^T] + \mathbf{e}_u\mathbf{e}_u^T \in \mathcal{L}$. Since S is a zero forcing set, $\mathbf{e}_v\mathbf{e}_v^T \in \mathcal{L}$ for all $v \in \{1, \ldots, n\}$. Then by (9) and (10), $\mathbf{e}_u\mathbf{e}_v^T \in \mathcal{L}$ for all neighbors u and v. Finally, since G is connected, for every $x, y \in \{1 \ldots, n\}$, there is a path $(x, v_1, v_2, \ldots, v_k, y)$. Then $\mathbf{e}_x\mathbf{e}_y^T = [[\ldots[\mathbf{e}_x\mathbf{e}_{v_2}^T, \mathbf{e}_{v_2}\mathbf{e}_{v_3}^T], \ldots], \mathbf{e}_{v_k}\mathbf{e}_y^T] \in \mathcal{L}$. ∎

Applying Proposition 2.1 we obtain the next corollary.

Corollary 4.2: If G is a connected graph, $A \in \mathcal{H}_n(\mathbb{R})$ with $\mathcal{G}(A) = G$, and $S \subseteq V$ is a zero forcing set of G, then $\langle iA, \{i\mathbf{e}_j\mathbf{e}_j^T : j \in S\}\rangle_{[\cdot,\cdot]} = u(n)$ and the corresponding quantum system is controllable.

Note that the converse of Theorem 4.1 is false.

Example 4.3: Consider the path on four vertices P_4 with the vertices numbered in order. The set $\{\mathbf{e}_2\}$ is not a zero forcing set for P_4. However,

$$\widetilde{W}\left(A_{P_4}, \{\mathbf{e}_2\}\right) = \begin{bmatrix}0&1&0&2\\1&0&2&0\\0&1&0&3\\0&0&1&0\end{bmatrix}$$

and $\operatorname{rank}\widetilde{W}(A_{P_4}, \{\mathbf{e}_2\}) = 4$, so $\mathcal{L}(A_{P_4}, \{\mathbf{e}_2\}) = gl(n, \mathbb{R})$ by Theorem 3.6.

V. Conclusion

Motivated by the control and dynamics of systems modeled on networks both classical and quantum, we have established a connection between various tests of controllability, and with the notion of zero forcing in graph theory. Lie algebraic quantum controllability is necessary and sufficient for (classical) linear controllability of an associated system and both notions are implied by the zero forcing property of the associated set of vertices. Linear systems have a very well developed theory [17] and it is an open question to investigate to what extent this analogy can be further used to discover properties of quantum systems and systems on networks.

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On Linear Solutions of the Output Feedback Pole Assignment Problem

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***Abstract*—A new linear method for solving the output feedback pole assignment problem of linear systems is introduced, and new sufficient conditions are obtained.**

***Index Terms*— Linear system, linear method, output feedback, pole assignment.**

I. Introduction

The output feedback pole assignment problem for linear systems has been studied for over 30 years by many authors. Early contributions on the subject were obtained by, e.g., Davison and Wang [4] and Kimura [10]. One of the major contributions to the problem was the introduction of the Hermann-Martin curve [8], [13], [14], where each m-input, p-output system of McMillan degree n is identified with a holomorphic curve of degree n on Grassmannian $\mathrm{Grass}(m, m+p)$. By using the dominant morphism theorem, they obtained the necessary and sufficient condition $n \geq mp$ for the generic pole assignability over $\mathbb{C}$. The second breakthrough was made by Brockett and Byrnes [1]. They connected the problem with the classical Schubert calculus and showed that when $n = mp$, the number of solutions over $\mathbb{C}$ equals $\deg \mathrm{Grass}(m, m+p)$. By using their formulation, Wang [19], [20] proved that $n > mp$ is sufficient over $\mathbb{R}$. This, coupled with Willems and Hesselink's result [18] that $n = mp$ is not sufficient over $\mathbb{R}$, provides the best sufficient condition.

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The pole assignment problem is highly nonlinear. For example, for 4×4 nondegenerate system of degree 16, the degree of the pole assignment map is 24,024 [1]. Furthermore, in the current method [20] for $n = mp-1$, a solution is obtained by tracing the solution fiber through a singular point of the pole assignment map. Because of this, the solution is extreme sensitive and useless for practical control problems. On the other hand, when $\max(m,p) = n$, the problem becomes the state feedback problem which can be solved by linear algebra, and the solutions are robust. The question is: to what extend linear methods can be used to solve the problem? Kimura's sufficient condition $n \leq m+p-1$ [10] is well known for the existence of linear solutions. A less known improvement was given by Rosenthal in [15] where he proved (assume $p \geq m$)

$$n \leq p + \left\lfloor \frac{m}{1} \right\rfloor + \cdots + \left\lfloor \frac{m}{k} \right\rfloor - 1, \quad \text{where } k = \left\lfloor \frac{p}{m} \right\rfloor \tag{I.1}$$

is sufficient for the existence of linear solutions for generic systems. The purpose of this technical note is to explore the possibility of linear algebra solutions with less freedom requirement.

This technical note is inspired by Konigorski's paper [11] in which the whole solutions were obtained in several stages. At the first stage right eigenvectors are selected and part of the feedback law is computed. At the second stage left eigenvectors are selected and another part of the feedback law is computed.

The technical note is organized as follows. In Section II, Konigorski's method is formulated in a much simpler way. New results are given in Section III. One of the corollary is that except for $\min(m,p) = 1$ and $m = p = 2$, $n \leq m + p$ is sufficient for the existence of linear solutions for generic systems, which improves Kimura's condition.

II. Linear Partial Pole Assignment

If an output feedback law $u = Ky$ is applied to a linear system $\dot{x} = Ax + Bu$, $y = Cx$, and if we identify a polynomial with its nonzero constant multiples, it is well known [19], [20] that the closed loop characteristic polynomial can be represented by

$$\begin{aligned}\det(sI - A - BKC) &= \det\left(D_l(s) - N_l(s)K\right)\\ &= \det\left(D_r(s) - KN_r(s)\right)\end{aligned}$$

where $D_l^{-1}(s)N_l(s) = N_r(s)D_r^{-1}(s)$ are left respectively right coprime (polynomial) decompositions of the transfer function $C(sI - A)^{-1}B$. We have four representations:

$$\det(sI - A - BKC) = \det \hat{P}(s)M = \det\left[M, P(s)\right] \tag{II.1}$$

$$= \det \hat{M}P(s) = \det \begin{bmatrix} \hat{P}(s) \\ \hat{M} \end{bmatrix} \tag{II.2}$$

where $M = \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right]$, $\hat{M} = [-K, I]$, $P(s) = \left[\begin{smallmatrix} N_r(s) \\ D_r(s) \end{smallmatrix}\right]$ and $\hat{P}(s) = [D_l(s), -N_l(s)]$. The formulas are true for any (unimodular) column equivalent M, $P(s)$ and row equivalent $\hat{M}$, $\hat{P}$. Clearly M and $\hat{M}$ represent the same compensator, and P and $\hat{P}$ represent the same system, if and only if $\hat{M}M = 0$, $\hat{P}(s)P(s) \equiv 0$.

We first consider assigning real poles. From the 2nd formulation of (II.1) we can see that a number s_0 is a closed loop pole if the column space of M intersects the column spaces of $P(s_0)$ nontrivially. We say vectors $v_1, \ldots, v_a$ assign the poles $s_1, \ldots, s_b$ if the space spanned by $v_1, \ldots, v_a$ intersects each of the column spaces of $P(s_1), \ldots, P(s_b)$ nontrivially.

Using the idea of [11], we choose the M in several stages. We first select α many vectors $H_1 = [v_1, \ldots, v_\alpha]$, $\alpha \leq p$, to assign k_α poles $\{s_i : i = 1, \ldots, k_\alpha\}$. This can be done easily if $k_\alpha = \alpha$, just take one